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Extended Lorentz invariance and field theory

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Abstract. The role of the extended Lorentz group as an invariance of physical theories is re-examined. Contrary to common acceptance it is asserted that all known theories, including those that describe the weak interactions, exhibit invariance under the extended Lorentz group of frame transformations. The fundamental discrete transformations are defined, independent of any particular field system, as endormorphisms on the space of spin frames over space–time. The appropriate transformations for spinor fields are considered in detail, and the notions of covariance under frame reversal is contrasted with other notions of 'parity invariance'.

1. Introduction

If we view local Lorentz covariance in terms of the active transformations that relate local observer frames of reference, then there seems little to prevent different observers employing natural spatial frames that are incoherently oriented with respect to each other. The description of phenomena from oppositely oriented frames is usually regarded in terms of the concept of parity. In certain situations global Cartesian spatial frames have been used to visualise parity operations on classical configurations. These ideas are formalised in most discussions on parity in terms of reflecting the global Cartesian coordinate system about some inconsequential origin. However, local orthonormal frame reversals, which make no reference to any particular system of coordinates, may be defined naturally on the bundle of orthonormal linear frames (Choquet-Bruhat *et al* 1970). Besides generalising the concept of parity to arbitrary space–times this has the virtue of removing the concept from dependence on any particular labelling of events in space–time. Although we shall not make any explicit reference to the gravitational field in this paper it is clearly desirable to extricate the notion of the extended Lorentz structure group from any particular geometry of space–time. Aside from these aesthetic considerations, we shall find that restricting the action of improper transformations to frame reversals puts a number of conventional arguments concerning the discrete transformations in field theory into a new perspective.

It is often stated that all relativistic field theories should be covariant under the proper orthochronous Lorentz group, but that they need not necessarily be covariant under the extended Lorentz group. In particular, it is commonly assumed that the theory of weak interactions does not have this extended covariance. In this paper we seek to clarify the role of covariance under the extended Lorentz group, in particular invariance under parity transformations. We discuss how familiar theories are fully covariant under the operation of frame reversal, and suggest that any asymmetries they

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possess should be delineated from their behaviour under such re-orientations of the space–time frames of reference.

There are certain unsatisfactory aspects of many treatments of the spinor transformation properties under the discrete transformations of the Lorentz group. One unsatisfactory feature is the consideration of a specific theory for the definition of these transformations. For example, properties of spinors under the Lorentz group are often found by looking for transformations which leave the Dirac equation invariant. It is subsequently concluded that Weyl's theory of the free neutrino lacks invariance under transformations that leave the Dirac equation invariant. It is thus sometimes said (see, for example, Berestetskiĭ *et al* 1971) that Weyl's theory lacks invariance under parity, unlike Majorana's theory. Other authors (Case 1957, McLennan 1957, Serpe 1957) have shown the equivalence of the two theories, and thus concluded that Weyl's theory is invariant if we amend the transformations on the Weyl spinors. More recent authors have adopted an ambivalent attitude to this situation (Marshak *et al* 1969, Sakurai 1964), maintaining that the important question is whether or not interactions are invariant under parity. A common resolution of the situation is to interpret the invariance of the Weyl equation as that produced by the operation of *CP*, where *C* is charge conjugation, and it is even suggested that *CP* should be relabelled *P* (Landau 1957). Various other proposals have also been made (Chang 1966, Werle 1973). Lee and Wick (1966) have commented on the unsatisfactory nature of the 'fuzziness' in the definitions of these discrete transformations, and claim that they are not even well defined unless they are such as to leave the theory invariant (Lee 1967). One of the prime motivations of this paper is to establish well defined discrete transformations independent of any particular theory. Furthermore, we shall argue that the extended local Lorentz group is a good symmetry for all known physical theories.

Properties of the basic spinor representations of the Lorentz group will be presented *ab initio*; partly to introduce the necessary notation, but also to emphasise that matrix methods are not always the most useful in the description of fermion fields. By formulating the action of the Lorentz group and its extension on the fields of the theory; in terms of linear transformations on spin frames which induce linear transformations on linear frames that span the tangent space of space–time; the discussion may be readily adapted to the case where the fields are operators on a Hilbert space of quantum states. It will not be necessary to draw attention to the induced transformations in Hilbert space, and we ignore the operational properties of the quantum fields. In particular, for simplicity we do not consider here anticommuting spinor fields.

The space–time manifold M is endowed with a pseudo-Riemannian metric tensor, g_M, with signature $(-1, 1, 1, 1)$. This metric allows the construction of g_M-orthonormal frames in the tangent space at each point of M. The local Lorentz group, L, can be defined as the group of all transformations that take one orthonormal frame to another orthonormal frame at the same point of M. Those elements of the group continuously connected to the identity form the proper orthochronous Lorentz group, L_0. The full Lorentz group L is made up of L_0, PL_0, TL_0 and PTL_0, where P and T are represented in the cotangent frame $\{e^0, e^1, e^2, e^3\}$ by

$$\mathrm{P}e^0 = e^0 \qquad \mathrm{P}e^k = -e^k \qquad \mathrm{T}e^0 = -e^0 \qquad \mathrm{T}e^k = e^k \qquad k = 1, 2, 3. \tag{1.1}$$

The basis vectors of the cotangent frame are labelled in accordance with the metric

$$g_M = -e^0 \otimes e^0 + \sum_{k=1}^{3} e^k \otimes e^k. \tag{1.2}$$

Theories may be constructed out of various fields, whose transformations under L are prescribed, by constructing an action density 4-form on M. Field equations can then be generated by an appropriate variational principle. A theory will be deemed Lorentz invariant if this action is a scalar under L. A theory such as free electromagnetism can be formulated solely in terms of L-scalar forms generated from the U(1) connection 1-form A pulled back to M. As the frames are changed under the L-group action the components of A transform contragradiently. Other theories involve fields which take a non-trivial representation of L (or its covering). For example, theories of gravity involve the curvature and torsion forms which transform non-trivially under L (Benn *et al* 1980). A field which takes a non-trivial representation of L is referred to here as a spin tensor.

L_0 is covered by SL(2, C), and a description of fermions involves fundamental representations of SL(2, C). It is important to remember that spinors do not provide a strict representation of L. So in discussing the discrete symmetries of P and T on spinors we are not looking for a representation of L; rather we seek a natural extension to SL(2, C) such that the tensor representations of this extended group, ISL(2, C), transform like tensors under L. To accord with customary terminology, however, we shall refer to spinor representations of the extended Lorentz group when dealing with representations of ISL(2, C).

If we have a Minkowskian space–time we can choose coordinates x^μ ($\mu = 0, 1, 2, 3$) such that $e^\mu = dx^\mu$. We can then define a diffeomorphism, π, from M to M, such that points with coordinates (x^0, x^i) are sent to points with coordinates $(x^0, -x^i)$. This will induce a transformation on tensors on M. In particular, on the field of Minkowski frames π^*: $(e^0, e^i) \to (e^0, -e^i)$. The usual 'orbital' parity is obtained from the behaviour of the tensor components of a particular solution under this diffeomorphism. This notion of parity, with its dependence on particular solutions and their behaviour over a region of M, must be carefully distinguished from the covariant description of events in M by observers who may choose differently oriented frames.

For completeness we also distinguish our frame parity from notions of handedness implied by the nature of any non-orientability of the space–time manifold. Such global complications can be incorporated into the discussion with the aid of twisted forms that incorporate a 'twist parity' that compensates for the change of sign of the coordinate transformation Jacobian necessary when integration is performed over a non-orientable manifold. Our subsequent discussion will be conducted in local neighbourhoods of the space–time manifold and any global consideration will not be relevant.

2. SL(2, C) spinors

Let $\{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ be a basis for a complex two-dimensional vector space V which carries the basic representation of SL(2, C). Denoting the six generators of SO(3, 1) (regarded as a real Lie algebra) by T^i $i = 1, \ldots, 6$ we write

$$T^i \boldsymbol{b}_\alpha = M^i_{\beta\alpha} \boldsymbol{b}_\beta \qquad \alpha, \beta = 1, 2 \tag{2.1}$$

where the six matrices $M^i = \{M^1, M^2, M^3, iM^1, iM^2, iM^3\}$ are given by

$$M^1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \qquad M^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad M^3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}. \tag{2.2}$$

The algebra of these matrices is isomorphic to the quaternion algebra.

If $\phi \in V$

$$\phi = \phi^{\alpha} \boldsymbol{b}_{\alpha} \xrightarrow{\mathrm{T}^i} \phi^{\alpha} \mathrm{T}^i \boldsymbol{b}_{\alpha} = \phi^{\alpha} M^i_{\beta\alpha} \boldsymbol{b}_{\beta} \tag{2.3}$$

and we induce the component transformations

$$\phi^{\alpha} \xrightarrow{\mathrm{T}^i} M^i_{\alpha\beta} \phi^{\beta}.$$

This complex 2D space can be endowed with a symplectic metric, g_v (i.e. a non-degenerate, antisymmetric, bilinear function) whose values are preserved under the group

$$g_v(\phi, \eta) = g_v(\mathrm{L}_0\phi, \mathrm{L}_0\eta) \tag{2.4}$$

since $\det \mathrm{L}_0 = 1$.

In this basis we specify its components to be

$$(g_{\alpha\beta}) = (g_v(\boldsymbol{b}_{\alpha}, \boldsymbol{b}_{\beta})) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.5}$$

and then

$$g_v(\eta, \xi) = \eta^{\alpha} \xi^{\beta} g_v(\boldsymbol{b}_{\alpha}, \boldsymbol{b}_{\beta}) = \eta^{\alpha} g_{\alpha\beta} \xi^{\beta} = \eta^{\alpha} \xi_{\alpha} \tag{2.6}$$

where it is convenient to use the lowering convention $\xi_{\alpha} = g_{\alpha\beta}\xi^{\beta}$.

Let V^* be the dual space to V: i.e. the space of linear functions of vectors in V, with values in C.

A basis for V^* is $\{\boldsymbol{b}^{\alpha}\}$:

$$\boldsymbol{b}^{\alpha}(\boldsymbol{b}_{\beta}) = \delta^{\alpha}_{\beta} \tag{2.7}$$

$\{\boldsymbol{b}^{\alpha}\}$ carries the contragradient representation

$$\mathrm{T}^i \boldsymbol{b}^{\alpha} = -M^i_{\alpha\beta} \boldsymbol{b}^{\beta}. \tag{2.8}$$

This ensures that

$$(Q\boldsymbol{b}^{\alpha})(Q\boldsymbol{b}_{\beta}) = \boldsymbol{b}^{\alpha}(\boldsymbol{b}_{\beta}) = \delta^{\alpha}_{\beta}\ \forall Q \in \mathrm{SL}(2, C). \tag{2.9}$$

We can use the (preserved) symplectic metric to associate vectors in V^* with vectors in V. Define $\tilde{\xi} \in V^*$ by

$$\tilde{\xi}(\eta) = g_v(\eta, \xi) \qquad \eta, \xi \in V. \tag{2.10}$$

If $\tilde{\xi} = \tilde{\xi}_{\alpha} \boldsymbol{b}^{\alpha}$ then $\tilde{\xi}(\eta) = \tilde{\xi}_{\alpha}\eta^{\alpha}$ $\forall \eta \in V$ and from (2.6) $\tilde{\xi}_{\alpha} = \xi_{\alpha}$.

Given the representation characterised by the matrices $\{M^i\}$, we always have another characterised by the complex conjugate. Let $\{\boldsymbol{c}_{\alpha}\}$ be a basis for a vector space X where

$$\mathrm{T}^i \boldsymbol{c}_{\alpha} = M^{i*}_{\beta\alpha} \boldsymbol{c}_{\beta}. \tag{2.11}$$

If $\{\boldsymbol{c}^{\alpha}\}$ is a basis for $X^*: \boldsymbol{c}^{\alpha}(\boldsymbol{c}_{\beta}) = \delta^{\alpha}_{\beta}$ then

$$\mathrm{T}^i \boldsymbol{c}^{\alpha} = -M^{i*}_{\alpha\beta} \boldsymbol{c}^{\beta}. \tag{2.12}$$

We can introduce a preserved symplectic metric on X, g_x, and associate vectors in X with vectors in X^* exactly as elements in V and V^* are related. It is customary to write $\phi \in X$ as $\phi = \phi^{\dot{\alpha}} \boldsymbol{c}_{\alpha}$. Of course this notation is redundant here as the basis indicates how the components transform.

For the real algebra $\{\mathrm{T}^i\}$ the conjugate representation is not an equivalent representation. However, for the subalgebra that generates SU(2), complex conjugate representations are equivalent and there exists a basis for V that transforms like $\{\boldsymbol{c}_\alpha\}$ under the SU(2) subalgebra. Since $Q^{\mathrm{T}} g_v Q = g_v \ \forall Q \in \mathrm{SL}(2, C)$ then $\boldsymbol{f}_\beta = g^{-1}_{\alpha\beta} \boldsymbol{b}_\alpha$ transforms like $\boldsymbol{c}_\beta$ under SU(2). Similarly $\boldsymbol{g}_\beta = g^{-1}_{\alpha\beta} \boldsymbol{c}_\alpha$ transforms like $\boldsymbol{b}_\beta$ under SU(2).

The situation is summarised in the following diagram.

V		X
basis $\{\boldsymbol{b}_\alpha\}$: $\mathrm{T}^i \boldsymbol{b}_\alpha = M^i_{\beta\alpha} \boldsymbol{b}_\beta$ basis $\{\boldsymbol{f}_\beta\}$: $\boldsymbol{f}_\beta = g^{-1}_{\alpha\beta} \boldsymbol{b}_\alpha$ $\mathrm{T}^i \boldsymbol{f}_\beta = M^{i*}_{\alpha\beta} \boldsymbol{f}_\alpha$ for $\mathrm{T}^i \in \mathscr{L}\mathrm{SU}(2)$ $\phi = \phi^\alpha \boldsymbol{b}_\alpha = \phi_\alpha \boldsymbol{f}_\alpha$	$\longleftrightarrow$ (complex) conjugate representations	basis $\{\boldsymbol{c}_\alpha\}$: $\mathrm{T}^i \boldsymbol{c}_\alpha = M^{i*}_{\beta\alpha} \boldsymbol{c}_\beta$ basis $\{\boldsymbol{g}_\beta\} = g^{-1}_{\alpha\beta} \boldsymbol{c}_\alpha$ $\mathrm{T}^i \boldsymbol{g}_\beta = M^i_{\alpha\beta} \boldsymbol{g}_\alpha$ for $\mathrm{T}^i \in \mathscr{L}\mathrm{SU}(2)$ $\eta = \eta^{\dot\alpha} \boldsymbol{c}_\alpha = \eta_{\dot\alpha} \boldsymbol{g}_\alpha$
V^* $\updownarrow$ dual vector *spaces*		X^* $\updownarrow$ dual vector *spaces*
basis $\{\boldsymbol{b}^\alpha\}$: $\boldsymbol{b}^\alpha(\boldsymbol{b}_\beta) = \delta^\alpha_\beta$ $\mathrm{T}^i \boldsymbol{b}^\alpha = -M^i_{\alpha\beta} \boldsymbol{b}^\beta$ $\boldsymbol{f}^\beta = g^{-1}_{\alpha\beta} \boldsymbol{b}^\alpha$ $\boldsymbol{f}^\beta(\boldsymbol{f}_\alpha) = \delta^\beta_\alpha$	(complex) conjugate $\longleftrightarrow$ representations	basis $\{\boldsymbol{c}^\alpha\}$: $\mathrm{T}^i \boldsymbol{c}^\alpha = -M^{i*}_{\alpha\beta} \boldsymbol{c}^\beta$ $\boldsymbol{g}^\beta = g^{-1}_{\alpha\beta} \boldsymbol{c}^\alpha$ $\boldsymbol{g}^\beta(g_\alpha) = \delta^\beta_\alpha$.

3. Parity

The vector space V is a representation space for SL(2, C), i.e. the group is realised by linear transformations from V to V. Can we introduce a linear transformation, P, from V to V such that $\mathrm{P}Q = Q\mathrm{P}$ iff $Q \in \mathrm{SU}(2)$? Equivalently, is there a matrix with these properties? The answer is no. However we can define a linear transformation P on the space $V \oplus X$ with these properties. Let

$$\mathrm{P}\boldsymbol{b}_\alpha = \boldsymbol{g}_\alpha \qquad \mathrm{P}\boldsymbol{c}_\alpha = \boldsymbol{f}_\alpha. \tag{3.1}$$

We obviously have the right commutation properties. This definition gives

$$\mathrm{P}\boldsymbol{f}_\alpha = -\boldsymbol{c}_\alpha \qquad \mathrm{P}\boldsymbol{g}_\alpha = -\boldsymbol{b}_\alpha \tag{3.2}$$

and so $\mathrm{P}^2 = -1$ on spinors.

We similarly define

$$\mathrm{P}\boldsymbol{b}^\alpha = \boldsymbol{g}^\alpha \qquad \mathrm{P}\boldsymbol{c}^\alpha = \boldsymbol{f}^\alpha \tag{3.3}$$

such that $(\mathrm{P}\boldsymbol{b}^\alpha)(\mathrm{P}\boldsymbol{b}_\beta) = \boldsymbol{b}^\alpha(\boldsymbol{b}_\beta) = \delta^\alpha_\beta$, etc.

As well as an arbitrary phase, the choice $\mathrm{P}^2 = -1$ is conventional, and we could have chosen $\mathrm{P}^2 = +1$. This point will be commented on later.

The vector space $S = V \oplus X$ is a representation space for the Lorentz group extended to include the mapping P. Let $\Phi \in S: \Phi = \phi^\alpha \boldsymbol{b}_\alpha + \eta_{\dot\alpha} \boldsymbol{g}_\alpha = \phi_\alpha \boldsymbol{f}_\alpha + \eta^{\dot\alpha} \boldsymbol{c}_\alpha$ then $\mathrm{P}\Phi = \phi^\alpha \boldsymbol{g}_\alpha - \eta_{\dot\alpha} \boldsymbol{b}_\alpha - \phi_\alpha \boldsymbol{c}_\alpha + \eta^{\dot\alpha} \boldsymbol{f}_\alpha$, and so we may induce the component transformation:

$$\phi^\alpha \xrightarrow{\mathrm{P}} -\eta_{\dot\alpha} \qquad \phi_\alpha \xrightarrow{\mathrm{P}} \eta^{\dot\alpha}$$

$$\eta_{\dot\alpha} \xrightarrow{\mathrm{P}} \phi^\alpha \qquad \eta^{\dot\alpha} \xrightarrow{\mathrm{P}} -\phi_\alpha. \tag{3.4}$$

Note however that if we write $\Phi = \phi \oplus \eta$, $\phi \in V$, $\eta \in X$ then P does not yield $\phi \to \eta$.

Both V and X have been given a metric which is preserved under SL(2, C). These metrics can be used to define a metric on S that is invariant under SL(2, C) and P.

Define $\mathrm{G}: S \times S \to C$ by

$$\mathrm{G}(\boldsymbol{b}_\alpha, \boldsymbol{b}_\beta) = g_v(\boldsymbol{b}_\alpha, \boldsymbol{b}_\beta) \qquad \mathrm{G}(\boldsymbol{c}_\alpha, \boldsymbol{c}_\beta) = g_x(\boldsymbol{c}_\alpha, \boldsymbol{c}_\beta) \qquad \mathrm{G}(\boldsymbol{b}_\alpha, \boldsymbol{c}_\beta) = 0$$
$$\mathrm{G}(\boldsymbol{c}_\alpha, \boldsymbol{b}_\beta) = 0 \qquad \forall \alpha, \beta. \tag{3.5}$$

It is readily verified that $\mathrm{G}(\mathrm{P}\Phi, \mathrm{P}\Lambda) = \mathrm{G}(\Phi, \Lambda) \forall \Phi, \Lambda \in S$.

4. Spin tensors

We define tensors on S and induce their transformation properties from the transformation of the basis of S.

Let

$$\{\boldsymbol{B}_\mu\} = \{\boldsymbol{b}_\alpha\} \cup \{\boldsymbol{c}_\beta\}$$
$$\{\boldsymbol{B}^\mu\} = \{\boldsymbol{b}^\alpha\} \cup \{\boldsymbol{c}^\beta\} \qquad \mu = 1, 2, 3, 4 \qquad \alpha, \beta = 1, 2. \tag{4.1}$$

An arbitrary element of $\mathcal{T}^1_1(S)$ is $\mathrm{T}^\mu_\nu \boldsymbol{B}_\mu \otimes \boldsymbol{B}^\nu$. $\mathcal{T}^1_1(S)$ is a 16D representation space of the extended group. This can be decomposed by the usual Clebsch–Gordan procedure into a sum of irreducible representations, i.e.

$$\mathcal{T}^1_1(S) = \mathcal{V} \oplus \mathcal{A} \oplus \mathcal{S} \oplus \mathcal{P} \oplus \Sigma \tag{4.2}$$

where $\mathcal{V}$ and $\mathcal{A}$ are 4D subspaces of $\mathcal{T}^1_1(S)$, $\mathcal{S}$ and $\mathcal{P}$ are 1D subspaces of $\mathcal{T}'_1(S)$ and Σ is a 6D subspace of $\mathcal{T}^1_1(S)$.

It can be checked that a basis for $\mathcal{V}$ is $\{\boldsymbol{q}_\mu\}$, where

$$\boldsymbol{q}_0 = -\boldsymbol{b}_1 \otimes \boldsymbol{c}^2 + \boldsymbol{c}_2 \otimes \boldsymbol{b}^1 - \boldsymbol{c}_1 \otimes \boldsymbol{b}^2 + \boldsymbol{b}_2 \otimes \boldsymbol{c}^1$$
$$\boldsymbol{q}_1 = \boldsymbol{b}_2 \otimes \boldsymbol{c}^2 + \boldsymbol{c}_2 \otimes \boldsymbol{b}^2 - \boldsymbol{b}_1 \otimes \boldsymbol{c}^1 - \boldsymbol{c}_1 \otimes \boldsymbol{b}^1$$
$$\boldsymbol{q}_2 = \mathrm{i}\{\boldsymbol{b}_1 \otimes \boldsymbol{c}^1 - \boldsymbol{c}_1 \otimes \boldsymbol{b}^1 + \boldsymbol{b}_2 \otimes \boldsymbol{c}^2 - \boldsymbol{c}_2 \otimes \boldsymbol{b}^2\}$$
$$\boldsymbol{q}_3 = \boldsymbol{b}_1 \otimes \boldsymbol{c}^2 + \boldsymbol{c}_2 \otimes \boldsymbol{b}^1 + \boldsymbol{b}_2 \otimes \boldsymbol{c}^1 + \boldsymbol{c}_1 \otimes \boldsymbol{b}^2 \tag{4.3}$$

and that $\mathrm{P}\boldsymbol{q}_0 = \boldsymbol{q}_0$, $\mathrm{P}\boldsymbol{q}_i = -\boldsymbol{q}_i$.

Furthermore we may use this basis to constuct a spin tensor valued coframe 1-form $e = e^\mu \boldsymbol{q}_\mu$ since P induces on the real one-forms the appropriate frame reversals under P.

A basis for $\mathcal{A}$ is $\{\eta_\mu\}$, where

$$\boldsymbol{\eta}_0 = -\boldsymbol{b}_1 \otimes \boldsymbol{c}^2 + \boldsymbol{c}_1 \otimes \boldsymbol{b}^2 + \boldsymbol{b}_2 \otimes \boldsymbol{c}^1 - \boldsymbol{c}_2 \otimes \boldsymbol{b}^1$$
$$\boldsymbol{\eta}_1 = \boldsymbol{b}_2 \otimes \boldsymbol{c}^2 + \boldsymbol{c}_1 \otimes \boldsymbol{b}^1 - \boldsymbol{b}_1 \otimes \boldsymbol{c}^1 - \boldsymbol{c}_2 \otimes \boldsymbol{b}^2$$
$$\boldsymbol{\eta}_2 = \mathrm{i}\{\boldsymbol{b}_1 \otimes \boldsymbol{c}^1 + \boldsymbol{c}_2 \otimes \boldsymbol{b}^2 + \boldsymbol{b}_2 \otimes \boldsymbol{c}^2 + \boldsymbol{c}_1 \otimes \boldsymbol{b}^1\}$$
$$\boldsymbol{\eta}_3 = \boldsymbol{b}_1 \otimes \boldsymbol{c}^2 + \boldsymbol{b}_2 \otimes \boldsymbol{c}^1 - \boldsymbol{c}_1 \otimes \boldsymbol{b}^2 - \boldsymbol{c}_2 \otimes \boldsymbol{b}^1. \tag{4.4}$$

It can be checked that $\mathrm{P}\boldsymbol{\eta}_0 = -\boldsymbol{\eta}_0$, $\mathrm{P}\boldsymbol{\eta}_i = \boldsymbol{\eta}_i$. The $\{\boldsymbol{\eta}_\mu\}$ are the appropriate spin tensor basis for the 3-form basis

$$e_{(3)} = e^1 \wedge e^2 \wedge e^3 \boldsymbol{\eta}_0 + e^0 \wedge e^2 \wedge e^3 \boldsymbol{\eta}_1 + e^0 \wedge e^3 \wedge e^1 \boldsymbol{\eta}_2 + e^0 \wedge e^1 \wedge e^2 \boldsymbol{\eta}_3. \tag{4.5}$$

A basis for $\mathscr{S}$ is

$$s = \boldsymbol{b}_1 \otimes \boldsymbol{b}^1 + \boldsymbol{b}_2 \otimes \boldsymbol{b}^2 + \boldsymbol{c}_1 \otimes \boldsymbol{c}^1 + \boldsymbol{c}_2 \otimes \boldsymbol{c}^2 \tag{4.6}$$

and $\mathrm{P}s = s$, $Qs = s$, $\forall Q \in \mathrm{SL}(2, C)$.

A basis for $\mathscr{P}$ is

$$\boldsymbol{p} = \boldsymbol{b}_1 \otimes \boldsymbol{b}^1 + \boldsymbol{b}_2 \otimes \boldsymbol{b}^2 - \boldsymbol{c}_1 \otimes \boldsymbol{c}^1 - \boldsymbol{c}_2 \otimes \boldsymbol{c}^2 \tag{4.7}$$

and $\mathrm{P}\boldsymbol{p} = -\boldsymbol{p}$, $Q\boldsymbol{p} = \boldsymbol{p}\ \forall Q \in \mathrm{SL}(2, C)$.

A basis for Σ is $\{\boldsymbol{\sigma}_i\}$, $i = 1, \ldots, 6$, where

$$\begin{aligned}
\boldsymbol{\sigma}_1 &= \boldsymbol{c}_2 \otimes \boldsymbol{c}^1 + \boldsymbol{b}_1 \otimes \boldsymbol{b}^2 + \boldsymbol{b}_2 \otimes \boldsymbol{b}^1 + \boldsymbol{c}_1 \otimes \boldsymbol{c}^2 \\
\boldsymbol{\sigma}_2 &= i\{\boldsymbol{b}_2 \otimes \boldsymbol{b}^1 + \boldsymbol{c}_1 \otimes \boldsymbol{c}^2 - \boldsymbol{c}_2 \otimes \boldsymbol{c}^1 - \boldsymbol{b}_1 \otimes \boldsymbol{b}^2\} \\
\boldsymbol{\sigma}_3 &= \boldsymbol{b}_1 \otimes \boldsymbol{b}^1 - \boldsymbol{b}_2 \otimes \boldsymbol{b}^2 + \boldsymbol{c}_1 \otimes \boldsymbol{c}^1 - \boldsymbol{c}_2 \otimes \boldsymbol{c}^2 \\
\boldsymbol{\sigma}_4 &= i\{\boldsymbol{b}_2 \otimes \boldsymbol{b}^2 + \boldsymbol{c}_1 \otimes \boldsymbol{c}^1 - \boldsymbol{b}_1 \otimes \boldsymbol{b}^1 - \boldsymbol{c}_2 \otimes \boldsymbol{c}^2\} \\
\boldsymbol{\sigma}_5 &= \boldsymbol{c}_2 \otimes \boldsymbol{c}^1 - \boldsymbol{b}_1 \otimes \boldsymbol{b}^2 + \boldsymbol{b}_2 \otimes \boldsymbol{b}^1 - \boldsymbol{c}_1 \otimes \boldsymbol{c}^2 \\
\boldsymbol{\sigma}_6 &= i\{\boldsymbol{c}_2 \otimes \boldsymbol{c}^1 - \boldsymbol{b}_1 \otimes \boldsymbol{b}^2 - \boldsymbol{b}_2 \otimes \boldsymbol{b}^1 + \boldsymbol{c}_1 \otimes \boldsymbol{c}^2\}.
\end{aligned} \tag{4.8}$$

If we put

$$e_{(2)} = e^0 \wedge e^1 \boldsymbol{\sigma}_1 + e^0 \wedge e^2 \boldsymbol{\sigma}_2 + e^0 \wedge e^3 \boldsymbol{\sigma}_3 + e^1 \wedge e^2 \boldsymbol{\sigma}_4 + e^3 \wedge e^1 \boldsymbol{\sigma}_5 + e^2 \wedge e^3 \boldsymbol{\sigma}_6 \tag{4.9}$$

where $e_{(2)}$ is the spin tensor valued 2-form basis, then we induce the appropriate transformations on the components.

Since we have a metric on S we can induce a metric on $\mathscr{T}^1_1(S)$. (The same symbol will be used for the induced metric.)

We define

$$\mathrm{G}(\boldsymbol{B}_\lambda \otimes \boldsymbol{B}^\rho, \boldsymbol{B}_\mu \otimes \boldsymbol{B}^\nu) = \mathrm{G}(\boldsymbol{B}_\lambda, \boldsymbol{B}_\mu) G(\boldsymbol{B}^\rho, \boldsymbol{B}^\nu). \tag{4.10}$$

It can be checked that, for example,

$$\mathrm{G}(\boldsymbol{q}_0, \boldsymbol{q}_0) = 4 \qquad \mathrm{G}(\boldsymbol{q}_i, \boldsymbol{q}_i) = -4 \qquad \text{and all others zero.}$$

5. Time reversal

So far we have extended $\mathrm{SL}(2, C)$ to include P, in such a way that we can identify the tensor product representations with representations of L_0 extended to include P. We now look for a transformation $\mathrm{T}: S \to S$, that allows us to continue this identification. That is, we seek

$$\mathrm{T}\boldsymbol{q}_\mu = -\mathrm{P}\boldsymbol{q}_\mu \qquad \mathrm{T}\boldsymbol{\eta}_\mu = -\mathrm{P}\boldsymbol{\eta}_\mu \qquad \mathrm{T}s = \mathrm{P}s \qquad \mathrm{T}\boldsymbol{p} = \mathrm{P}\boldsymbol{p} \qquad \mathrm{T}\boldsymbol{\sigma}_i = \mathrm{P}\boldsymbol{\sigma}_i \tag{5.1}$$

This requires that

$$\begin{aligned}
&\mathrm{T}(\boldsymbol{b}_\alpha \otimes \boldsymbol{c}^\beta) = -\mathrm{P}(\boldsymbol{b}_\alpha \otimes \boldsymbol{c}^\beta) \\
&\mathrm{T}(\boldsymbol{b}_\alpha \otimes \boldsymbol{b}^\beta) = \mathrm{P}(\boldsymbol{b}_\alpha \otimes \boldsymbol{b}^\beta) \\
&\mathrm{T}(\boldsymbol{c}_\alpha \otimes \boldsymbol{c}^\beta) = \mathrm{P}(\boldsymbol{c}_\alpha \otimes \boldsymbol{c}^\beta) \qquad \forall_{\alpha,\alpha}.
\end{aligned} \tag{5.2}$$

In terms of the basic space S we need either

$$\begin{aligned} \mathrm{T}\boldsymbol{b}_\alpha &= -\mathrm{P}\boldsymbol{b}_\alpha \\ \mathrm{T}\boldsymbol{c}_\alpha &= \mathrm{P}\boldsymbol{c}_\alpha \end{aligned} \qquad \text{or} \qquad \begin{aligned} \mathrm{T}\boldsymbol{b}_\alpha &= \mathrm{P}\boldsymbol{b}_\alpha \\ \mathrm{T}\boldsymbol{c}_\alpha &= -\mathrm{P}\boldsymbol{c}_\alpha. \end{aligned} \tag{5.3}$$

We choose the former, i.e.

$$\begin{aligned} \mathrm{T}\boldsymbol{b}_\alpha &= -\boldsymbol{g}_\alpha \qquad & \mathrm{T}\boldsymbol{c}_\alpha &= \boldsymbol{f}_\alpha \\ \mathrm{T}\boldsymbol{f}_\alpha &= \boldsymbol{c}_\alpha \qquad & \mathrm{T}\boldsymbol{g}_\alpha &= -\boldsymbol{b}_\alpha. \end{aligned} \tag{5.4}$$

It can be checked that $\mathrm{T}^2 = +1$, $\mathrm{TP} = -\mathrm{PT}$ on spinors, and $\mathrm{PT}\boldsymbol{b}_\alpha = \boldsymbol{b}_\alpha$, $\mathrm{PT}\boldsymbol{c}_\alpha = -\boldsymbol{c}_\alpha$. Further, $\mathrm{G}(\mathrm{T}\Phi, \mathrm{T}\Lambda) = \mathrm{G}(\Phi, \Lambda) \forall \Phi, \Lambda \in \boldsymbol{S}$, so G is preserved by the full Lorentz group.

We can now return to the question of the implications in choosing $\mathrm{P}^2 = +1$, or -1, on spinors. We could have adopted for our definition of P what is here called T, thus giving $\mathrm{P}^2 = +1$. Had we done so we could have re-identified ISL(2, C) irreducible representations in $\mathscr{T}^1_1(S)$ with L reresentations, and would then have looked for a $\mathrm{T}: \mathrm{T}\boldsymbol{b}_\alpha = -\mathrm{P}\boldsymbol{b}_\alpha, \mathrm{T}\boldsymbol{c}_\alpha = \mathrm{P}\boldsymbol{c}_\alpha$. We would thus have chosen for T what has here been called P. So we can choose $\mathrm{P}^2 = \pm 1$, but need $\mathrm{T}^2 = \mp 1$.

6. Charge conjugation

We define C to be a conjugate linear transformation that maps the basis of a representation space to the basis of the (complex) conjugate representation space (for *any* group $\mathscr{G}$).

If

$$\boldsymbol{B}_\alpha \xrightarrow{\mathscr{G}} Q_{\beta\alpha}\boldsymbol{B}_\beta \tag{6.1}$$

and

$$\boldsymbol{B}^c_\alpha \xrightarrow{\mathscr{G}} Q^*_{\beta\alpha}\boldsymbol{B}^c_\beta \tag{6.2}$$

then

$$\phi = \phi^\alpha \boldsymbol{B}_\alpha \xrightarrow{C} \phi^{\alpha *}\boldsymbol{B}^c_\alpha. \tag{6.3}$$

It can be seen from this definition that $C^2 = +1$, $[C, Q] = 0 \ \forall Q \in \mathscr{G}$ (for any $\mathscr{G}$). In particular, if $\mathscr{G}$ is SL(2, C), let

$$\Phi = \xi^\alpha \boldsymbol{b}_\alpha + \eta^{\dot\alpha}\boldsymbol{c}_\alpha \qquad \in S$$

then

$$C\Phi = \Phi_c = \xi^{\alpha *}\boldsymbol{c}_\alpha + \eta^{\dot\alpha *}\boldsymbol{b}_\alpha \tag{6.4}$$

and so we induce the component transformation

$$\xi^\alpha \to \eta^{\dot\alpha *} \qquad \eta^{\dot\alpha} \to \xi^{\alpha *}. \tag{6.5}$$

We can now check the commutation relations between T, C, and P

$$\begin{array}{ccc} \Phi = (\xi^\alpha \boldsymbol{b}_\alpha + \eta^{\dot\alpha}\boldsymbol{c}_\alpha) & \xrightarrow{C} & (\xi^{\alpha *}\boldsymbol{c}_\alpha + \eta^{\dot\alpha *}\boldsymbol{b}_\alpha) \\ \downarrow^{P} & & \downarrow^{P} \\ (\xi^\alpha \boldsymbol{g}_\alpha + \eta^{\dot\alpha}\boldsymbol{f}_\alpha) & \xrightarrow{C} & (\xi^{\alpha *}\boldsymbol{f}_\alpha + \eta^{\dot\alpha *}\boldsymbol{g}_\alpha). \end{array} \tag{6.6}$$

Hence $C\mathrm{P}=\mathrm{P}C$ on spinors. Similarly we can show $C\mathrm{T}=-\mathrm{T}C$. Note, however, that since C is conjugate linear, arbitrary phase factors can change these commutation relations.

It is readily verified that

$$\mathrm{G}(C\Phi, C\Lambda)=\mathrm{G}^*(\Phi, \Lambda) \qquad \forall\Phi, \Lambda\in S. \tag{6.7}$$

C is induced on $\mathscr{T}^p_q(S)$ in a natural manner

$$C(\boldsymbol{B}_\mu\otimes\boldsymbol{B}^\nu)=C\boldsymbol{B}_\mu\otimes C\boldsymbol{B}^\nu \tag{6.8}$$

ensuring that $[C, Q]=0\ \forall Q\in \mathrm{SL}(2, C)$.

It may also be verified that

$$\begin{aligned} &C\boldsymbol{q}_\mu=\boldsymbol{q}_\mu \qquad C\boldsymbol{s}=\boldsymbol{s} \\ &C\boldsymbol{\eta}_\mu=-\boldsymbol{\eta}_\mu \qquad C\boldsymbol{p}=-\boldsymbol{p} \qquad C\boldsymbol{\sigma}_i=\boldsymbol{\sigma}_i \end{aligned} \tag{6.9}$$

although no significance should be attached to the signs as they can be changed by modifying phase factors, e.g. $\boldsymbol{q}'_\mu\equiv \mathrm{i}\boldsymbol{q}_\mu : C\boldsymbol{q}'_\mu=C\mathrm{i}\boldsymbol{q}_\mu=-\mathrm{i}C\boldsymbol{q}_\mu=-\boldsymbol{q}'_\mu$.

7. Construction of a preserved Hermitian metric on *S*

We define a new metric G_{H} on S in terms of G and C:

$$\mathrm{G_H}(\Psi, \Lambda)=\mathrm{i}\mathrm{G}(C\Psi, \Lambda) \qquad \Psi, \Lambda\in S \tag{7.1}$$

$\mathrm{G_H}$ is obviously conjugate linear in the first variable and linear in the second. Also

$$\begin{aligned} \mathrm{G_H}(\Lambda, \Psi)&=\mathrm{i}\mathrm{G}(C\Lambda, \Psi) \\ &=\mathrm{i}\mathrm{G}^*(C^2\Lambda, C\Psi) \qquad \text{by (6.7)} \\ &=\mathrm{i}\mathrm{G}^*(\Lambda, C\Psi) \\ &=-\mathrm{i}\mathrm{G}^*(C\Psi, \Lambda) \\ &=\mathrm{G}^*_{\mathrm{H}}(\Psi, \Lambda) \end{aligned} \tag{7.2}$$

so $\mathrm{G_H}$ is Hermitian symmetric.

We can use this Hermitian metric to associate vectors in S with vectors in S^*. We define $\bar{\Psi}\in S^*$ by

$$\bar{\Psi}(\Lambda)=\mathrm{G_H}(\Psi, \Lambda) \qquad \forall\Lambda\in S. \tag{7.3}$$

Then

$$\bar{\Psi}(\Lambda)=\mathrm{i}\mathrm{G}(\Psi_c, \Lambda)=-\mathrm{i}\mathrm{G}(\Lambda, \Psi_c)=-\mathrm{i}\tilde{\Psi}_c(\Lambda) \tag{7.4}$$

so $\bar{\Psi}=-\mathrm{i}\tilde{\Psi}_c$. $\bar{\Psi}$ will be referred to as the Dirac adjoint, or Hermitian adjoint of Ψ.

This Hermitian metric will be used in forming real actions in later sections, and in extending this formalism to the case where we have a symmetry group $\mathrm{SL}(2, C)\times\mathscr{G}_{\mathrm{u}}$ for some unitary 'internal' group $\mathscr{G}_{\mathrm{u}}$.

8. Relation to matrix representations

Having made our definitions in generality we can now introduce matrix methods if we wish. We can adopt as a basis for S

$$b_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad b_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad c_1 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad c_2 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \tag{8.1}$$

Then the basis for S^* can be written as row vectors, and their action on S is matrix multiplication. Elements of $\mathscr{T}^1_1(S)$ are written as square matrices; their action on elements of S being multiplication from the left, and on elements of S^*, multiplication from the right.

To illuminate the role played by matrix multiplication we define the contracted tensor product map

$$\circ : \mathscr{T}^1_1(S) X \mathscr{T}^1_1(S) \to \mathscr{T}^1_1(S) \tag{8.2}$$

by $(A \circ B)(\Lambda, \psi) = [A(\Lambda)](B(\psi))$, $A, B \in \mathscr{T}^1_1(S)$ $\forall \Lambda \in S^*$, $\psi \in S$. It is easy to see that in matrix language

$$\Lambda \cdot (A \circ B) \cdot \psi = (\Lambda \cdot A) \cdot (B \cdot \psi) = \Lambda \cdot (A \cdot B) \cdot \psi$$

where $\cdot$ refers to matrix multiplication. Thus matrix multiplication of the matrix representations of $\mathscr{T}^1_1(S)$ corresponds to the contracted tensor product, $\circ$.

In the rest of this paper we shall simply write AB instead of $A \circ B$, for $A, B \in \mathscr{T}^1_1(S)$. We shall also simplify our notation by writing $A\psi$ instead of $A(\psi)$, $A \in \mathscr{T}^1_1(S)$, $\psi \in S$ and ΛA insted of $A(\Lambda)$, $A \in \mathscr{T}^1_1(S)$, $\Lambda \in S^*$. This notation is such that we may readily use matrix representations should we so wish.

To conform with convention we shall write the matrix representations of the $\{\boldsymbol{q}_\mu\}$ as $\{\gamma_\mu\}$. In the chosen basis

$$\begin{aligned} \gamma_0 &= \begin{pmatrix} 0 & M^2 \\ M^2 & 0 \end{pmatrix} \qquad & \gamma_1 &= \begin{pmatrix} 0 & -\mathrm{i}M^3 \\ -\mathrm{i}M^3 & 0 \end{pmatrix} \\ \gamma_2 &= \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} \qquad & \gamma_3 &= \begin{pmatrix} 0 & \mathrm{i}M^1 \\ \mathrm{i}M^1 & 0 \end{pmatrix} \end{aligned} \tag{8.3}$$

where the $\{M^i\}$ were introduced in (2.2).

We may similarly find the matrix representations of the other elements of $\mathscr{T}^1_1(S)$.

$$s = 1 \text{ (the } 4 \times 4 \text{ identity matrix)} \tag{8.4}$$

$$\boldsymbol{p} = \gamma_5 \equiv \mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3 \tag{8.5}$$

$$\boldsymbol{\eta}_\mu = \gamma_5\gamma_\mu \tag{8.6}$$

$$\begin{aligned} &\boldsymbol{\sigma}_1 = \gamma_1\gamma_0 \qquad \boldsymbol{\sigma}_2 = \gamma_2\gamma_0 \qquad \boldsymbol{\sigma}_3 = \gamma_3\gamma_0 \\ &\boldsymbol{\sigma}_4 = \gamma_2\gamma_1 \qquad \boldsymbol{\sigma}_5 = \gamma_1\gamma_3 \qquad \boldsymbol{\sigma}_6 = \gamma_3\gamma_2. \end{aligned} \tag{8.7}$$

Also

$$\mathrm{G} = \gamma_3\gamma_1 \tag{8.8}$$

and

$$\gamma_0^\dagger = -\gamma_0 \qquad \gamma_i^\dagger = \gamma_i. \tag{8.9}$$

It should be noted that † means 'conjugate transpose', and not Hermitian conjugation.

It can be verified that

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} \tag{8.10}$$

where $\eta_{\mu\nu}$ are the orthonormal components of g_M; either by matrix multiplication using (8.3); or by confirming that $\boldsymbol{q}_\mu \circ \boldsymbol{q}_\nu + \boldsymbol{q}_\nu \circ \boldsymbol{q}_\mu = 2\eta_{\mu\nu}$, using (4.3).

It is readily checked that

$$\mathrm{P}\psi = \gamma_0\psi \tag{8.11}$$

and that

$$\mathrm{T}\psi = \mathrm{i}\gamma_1\gamma_2\gamma_3\psi \tag{8.12}$$

corresponding to the definition introduced by Racah (1937)†. Also

$$C\psi = \gamma_1\gamma_0\gamma_3\psi^* \tag{8.13}$$

and

$$\bar{\psi} = \mathrm{i}\psi^\dagger\gamma_0. \tag{8.14}$$

9. The Weyl equation

The appearance of familiar expressions in the previous section might give the impression that we have so far merely presented well known facts, in a somewhat long-winded manner. However, the following simple example illustrates how working with the intrinsic spin tensors removes the ambiguity referred to in the introduction.

The Weyl equation can be written in this language as

$$*e \wedge \mathrm{d}\psi = 0 \tag{9.1}$$

where $\psi \in V \subset S$; $e \equiv e^\mu \boldsymbol{q}_\mu$, the spin tensor valued 1-form basis; d is the exterior derivative, and $*$ is the Hodge map $\Lambda^p(M) \to \Lambda^{4-p}(M)$. Here we have, for simplicity, taken a gauge in which the space–time connection, describing the gravitational field, is zero (see Benn *et al* (1980) for more generality).

Choosing global Minkowskian coordinates, x^μ, (9.1) can be written as

$$*e^\mu \boldsymbol{q}_\mu \wedge \partial_\nu\psi^\alpha e^\nu \boldsymbol{b}_\alpha = *e^\mu \wedge e^\nu \partial_\nu \psi^\alpha \boldsymbol{q}_\mu(\boldsymbol{b}_\alpha) = 0$$

$$-\eta^{\mu\nu} *1 \partial_\nu \psi^\alpha \boldsymbol{q}_\mu(\boldsymbol{b}_\alpha) = -\partial^\mu \psi^\alpha \boldsymbol{q}_\mu(\boldsymbol{b}_\alpha) *1 = 0.$$

So (9.1) gives

$$\partial^\mu \psi^\alpha \boldsymbol{q}_\mu(\boldsymbol{b}_\alpha) = 0. \tag{9.2}$$

In this form the covariance under parity is apparent; $\boldsymbol{q}_\mu(\boldsymbol{b}_\alpha) \in X$, whereas $(\mathrm{P}\boldsymbol{q}_\mu)\mathrm{P}(\boldsymbol{b}_\alpha) \in V$; but the components of the equation are *unaltered*. (In particular we do not take the complex conjugate! cf Berestetskiĭ *et al* (1971)).

† cf and contrast our approach and conclusions with Jauch and Rohrlick (1955).

Although ψ has only two non-vanishing components (9.1) should not be written as a 2×2 matrix equation. $\mathscr{V}\in\mathscr{T}^1_1(S)$ is irreducible under L: it is however reducible under L_0. These reduced parts of $\mathscr{V}$ can be represented by 2×2 matrices (the Pauli matrices), but we cannot then induce the appropriate transformations on e^{μ}. Thus the requirement that we induce the full Lorentz transformation properties of the e^{μ} from the transformation of their spin tensor basis fixes that basis for us, and ensures the covariance of (9.1) under L, independent of the number of non-vanishing components of ψ.

10. Unitary internal groups

In general the fields in a physical theory will be representations of $L\times\mathscr{G}_u$, where $\mathscr{G}_u$ is some 'internal' group, assumed here unitary. We show how the preceding formalism is easily extended to accommodate this situation.

Let U be a representation space for $\mathscr{G}_u$. Since $\mathscr{G}_u$ is unitary it allows a preserved Hermitian metric, G_u, to be defined on U; and we can choose a basis $\{\boldsymbol{u}_\alpha\}$ for U such that $G_u(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta)=\delta_{\alpha\beta}$. Let $\{\boldsymbol{u}^\alpha\}$, such that $\boldsymbol{u}^\alpha(\boldsymbol{u}_\beta)=\delta^\alpha_\beta$, be a basis for U^*. We can use G_u to associate vectors in U with vectors in U^*. We define $\chi^\dagger\in U^*$ by

$$\chi^\dagger(\lambda)=G_u(\chi,\lambda)\qquad \chi,\lambda\in U. \tag{10.1}$$

If $\chi=\chi^\alpha\boldsymbol{u}_\alpha$, then this gives $\chi^\dagger=\chi^{\alpha *}\boldsymbol{u}^\alpha$.

We define a Hermitian metric, G_T, on $S\otimes U$, which is preserved by $L\times\mathscr{G}_u$, by

$$G_T(\boldsymbol{B}_\lambda\otimes\boldsymbol{u}_\beta, \boldsymbol{B}_\mu\otimes\boldsymbol{u}_\nu)=G_H(\boldsymbol{B}_\lambda,\boldsymbol{B}_\mu)G_u(\boldsymbol{u}_\beta,\boldsymbol{u}_0). \tag{10.2}$$

The notion of the Hermitian adjoint is extended by defining $\bar\psi\in S^*\otimes U^*$ by

$$\bar\psi(\Lambda)=G_T(\psi,\Lambda)\qquad \psi,\Lambda\in S\otimes U. \tag{10.3}$$

11. A simple model

We now take an illustrative example of a simple model that is invariant under L, but is conventionally interpreted as not being invariant under parity.

Let U be a representation space for U(1); spanned by $\boldsymbol{u}$ such that $Q\boldsymbol{u}=\lambda\mathrm{i}\boldsymbol{u}$; Q being the U(1) generator, and λ some integer. Let W be some (in general distinct) representation space of the U(1) group, spanned by $\boldsymbol{w}$ such that $Q\boldsymbol{w}=\mu\mathrm{i}\boldsymbol{w}$, for some integer μ. Consider the action density 4-form on M:

$$\Lambda=2\,\mathrm{Re}\,(\bar\xi * e\wedge D\xi+\bar\eta * e\wedge D\eta)\qquad \xi\in V\otimes U\qquad \eta\in X\otimes W. \tag{11.1}$$

D is the U(1) gauge covariant derivative defined by

$$D\xi=\mathrm{d}\xi+\mathrm{i}\lambda A\xi\qquad D\eta=\mathrm{d}\eta+\mathrm{i}\mu A\eta \tag{11.2}$$

where A is the (real) U(1) connection 1-form. Λ is invariant under the extended Lorentz group, as may be seen by writing $\bar\xi * e\wedge D\xi$ as $G_T(\xi, *e\wedge D\xi)$ and recalling that G_H (and thus G_T) is invariant under all of L. Note that if $\xi\equiv 0$, $\mu=0$, (11.1) reduces to the free Weyl theory, and the preceding remarks emphasise its invariance under L.

In the general case writing out the covariant derivatives explicitly gives

$$\Lambda=2\,\mathrm{Re}\,(\bar\xi * e\wedge\lambda\mathrm{i}A\xi+\bar\eta * e\wedge\mathrm{i}\mu A\eta)+\text{kinetic terms} \tag{11.3}$$

from which we identify the interaction as $A \wedge J$ with

$$J = 2\ \mathrm{Im}\ (\lambda \bar{\xi} * e\xi + \mu \bar{\eta} * e\eta). \tag{11.4}$$

Consider first the particular case of $\mu = 0$. Since $\xi \in V$, $p\xi \equiv \xi$ (p spans $\mathscr{P} \in \mathscr{T}^1_1(S)$), and J may be written

$$J = \lambda\ \mathrm{Im}\ (\bar{\xi} * e\xi + \bar{\xi} * ep\xi) \tag{11.5}$$

This is traditionally called a '$V + A$' current, and its alleged transformation properties under L are the reasons usually given for the absence of an electrically charged neutrino (Gursey 1950, Serpe 1949). This argument depends on associating the behaviour of the conventional electric current under parity with that of the right hand side of Maxwell's source equations. These may be written as L-scalar 1-form equations

$$* \mathrm{d} * F = * J \tag{11.6}$$

where $F = \mathrm{d}A$.

If the U(1) invariance that generated the current in (11.5) were that of electromagnetism then (11.6) becomes

$$\begin{aligned}(* \mathrm{d} * F)_\mu e^\mu &= * \lambda\ \mathrm{Im}\ (\bar{\xi} * e\xi + \bar{\xi} * ep\xi) \\ &= \lambda\ \mathrm{Im}\ (\bar{\xi}\gamma_\mu \xi + \bar{\xi}\gamma_\mu\gamma_5\xi) e^\mu \end{aligned} \tag{11.7}$$

where we have used the matrix representation of the spin tensors. This may be written as a spin tensor valued 0-form equation

$$(* \mathrm{d} * F)_\mu \gamma^\mu = (* J)_\mu \gamma^\mu \tag{11.8}$$

where $\gamma^\mu \equiv \eta^{\mu\nu}\gamma_\nu$ such that $\gamma^\mu\gamma_\nu = \delta^\mu_\nu$.

The important fact is that this equation has been derived from an extended Lorentz invariant theory, so the absence of an electrically charged neutrino must be explained by reasons that are distinct from lack of invariance under L.

Consider the special case of (11.1) for which $\lambda = \mu$. We can then define $\psi \in S \otimes U$ by

$$\psi = \xi \oplus \eta \tag{11.9}$$

(if $\lambda = \mu$, U and W are isomorphic and we may identify them). (11.1) can then be written as

$$\begin{aligned}\Lambda &= 2\ \mathrm{Re}\ (\bar{\psi} * eD\psi) \\ &= -2\ \mathrm{Re}\ (\bar{\psi}\gamma_\mu D\psi) * e^\mu \end{aligned} \tag{11.10}$$

and it may be recognised as the kinetic part of the U(1) invariant Dirac action. (11.10) is invariant under the interchange of ξ and η; a transformation between a spinor in X and one in V with the same U(1) charge (as is the parity transformation). Note, however, that whereas (11.1) is invariant under L for all values of λ and μ, we have the above field transformations as a symmetry iff $\lambda = \mu$. Further, we see that this case generates what is conventionally called a 'pure V' current; that is

$$* J = 2\lambda\ \mathrm{Im}\ (\bar{\psi}\gamma_\mu\psi) e^\mu. \tag{11.11}$$

Thus for the model considered here the absence or presence of these field symmetries corresponds to what is conventionally called parity non-invariance and parity invariance.

12. Conclusion

We have demonstrated that a consistent definition of space–time frame reversal can be established by inducing on the orthonormal frame bundle, transformation defined fundamentally on a bundle of spin frames over M. These have been established to carry linear or conjugate linear representations of operations that have been identified as P, T and C. Identifying the components of vectors in these frames with the components of Lorentz group spinors we find that such operations are represented by conventional spinor component transformations under the extended Lorentz group to within an arbitrary set of phases. With the aid of ISL(2, C) invariant metrics we have shown how to construct action 4-forms on M that are invariant under this extended Lorentz group. The procedure has been naturally generalised to include C invariance with respect to additional unitary internal symmetry groups.

Our main conclusion is that the concept of so-called non-invariance of physical theories under the discrete transformation P should not be confused with the formulation of physical laws in oppositely oriented space–time reference frames. The notion of such symmetry violations must be sought in the behaviour of the dynamical field equations of motion under appropriate field transformations. We have already noted that conventional parity violating terms in an action are not invariant under the chiral interchange $\xi \leftrightarrow \eta$. We stress that this interchange of complete spin tensors is not the same as our transformation (3.2) under P. In the current theory of the electroweak processes (Salam 1968, Weinberg 1967) such an asymmetry is built into the theory of course by assigning different dynamical charges to the chiral lepton spinors. Similarly the pion is identified as a pseudoscalar field fundamentally because of the way it is found to couple to the nucleon current. Since the space–time frames themselves are legitimate dynamical fields we should allow transformations that involve the e^{α} among such dynamical field symmetries. Such transformations are at the heart of the dual operator $F \leftrightarrow *F$ that distinguishes between interactions such as $F \wedge F$ and $F \wedge *F$ involving gauge field curvatures F. In their component formulation such terms are often distinguished on the grounds of their behaviour under parity. With the formulation of theories in terms of extended L-group tensors we hope to have clarified the distinction between covariance under frame reversal and the asymmetries that have their origin in certain field transformations of interacting field systems.

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